PHYS393: Statistical and Low Temperature Physics

Tutorial 4

Useful constants:

Electron rest mass	m_e	$9.109 \times 10{-}31$ kg
Atomic mass unit	u	1.6605×10^{-27} kg
Boltzmann's constant	k_B	1.3807×10^{-23} J K^{-1}
Gas constant	R	8.314 J mol^{-1} K^{-1}
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
Bohr magneton	μ_{B}	9.274×10^{-24} J T^{-1}

1. At room temperature, normal H_2 contains 75% ortho-hydrogen and 25% para-hydrogen. When cooled down to 20 K (just below the boiling point), it would take some time before most of the ortho-hydrogen is converted to para-hydrogen.

   (i) Calculate the Boltzmann factors for the angular momentum $J = 0$, 1, 2 and 3. Is it reasonable to assume that only the ground state $J = 1$ in ortho-hydrogen, and the ground state $J = 0$ in para-hydrogen are occupied? Explain.

   (ii) Assuming that only the ground states are occupied, calculate the heat energy released when one mole of normal H_2 at 20 K is converted to one mole of para-H_2.

   (iii) Given that the latent heat is 0.94 kJ/mol, what would then happen to the liquid hydrogen?

   [You are given that the moment of inertia of the H_2 rotator is $I = 4.59 \times 10^{48}$ kg m^2.]

2. (i) For a reversible change, the entropy is related to the heat absorbed Q and the temperature T by

   $$S = \int \frac{dQ}{T}.$$

   Assuming that liquid helium-3 is a Fermi gas, show that the entropy per mole of liquid helium-3 is equal to its heat capacity

   $$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R,$$

   where T_F is the Fermi temperature and R is the gas constant.

   (ii) The helium-3 in the two phases in a dilution refrigerator are approximately in equlibrium. At phase equilibrium, the chemical potentials of the phases are equal:

   $$H_C - TS_C = H_D - TS_D,$$

   where H is the enthalpy per mole. The subscripts D and C denote the dilute phase (6.6% concentration) and the concentrated phase respectively.
   Show that the enthalpy change of mixing in the dilution refrigerator is given by

   $$\Delta H = T(C_D - C_C),$$

   where C_D and C_C are the heat capacities.

(iii) Given that the Fermi temperatures of the dilute phase and the concentrated phase are 0.4 K and 1.8 K respectively, show that the enthalpy change is

$$\Delta H = 80T^2 \text{ [J/mol]}.$$

3. Cerium Magnesium Nitrate (CMN) is a spin-1/2 paramagnetic salt that has been used for magnetic cooling. A sample of the salt is initially at a magnetic field B of 2 T and a temperature T of 1 K.

   (i) During demagnetisation, the applied field is reduced to zero. The remaining interaction field b is related to the magnetic ordering temperature T_c by

   $$\mu_B b = k_B T_c.$$

   Given that the ordering temperature of CMN is 2 mK, calculate the interaction field that remains, and hence the final temperature. (Hint: During demagnetisation, the temperature is proportional to the field.)

   (ii) The magnetic energy in a spin-1/2 salt is given by

   $$U = -N\mu_B B \tanh\left(\frac{\mu_B B}{k_B T}\right),$$

   where N is the number of particles. The cooling power of of the CMN is the heat that it would absorb when it warms up. Calculate the cooling power of one mole of the CMN.

   (iii) The magnetic entropy in a spin-1/2 salt is given by

   $$S = Nk_B \ln\left[2\cosh\left(\frac{\mu_B B}{k_B T}\right)\right] - \frac{N\mu_B B}{T}\tanh\left(\frac{\mu_B B}{k_B T}\right).$$

   Calculate for the same experiment the heat of magnetisation which has to be removed if this salt is magnetised isothermally to 2 T.